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REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the American Mathematical Society classification scheme. The 2000 Mathematics Subject Classification can be found in print starting with the 1999 annual index of *Mathematical Reviews.* The classifications are also accessible from `www.ams.org/msc/`.

2[65N30, 75V05, 76M10]—*p- and hp-finite element methods. Theory and applications in solid and fluid mechanics*, by Ch. Schwab, Oxford Science Publications, Clarendon Press, 1998, xii+374 pp., hardcover, $85.00

Over the past half century, the finite element method has emerged as the method of choice for the numerical approximation of elliptic boundary value problems, particularly those arising in structural mechanics and particularly amongst the engineering community. Although mathematicians like to cite the work of Courant as representing the beginnings of the method, the major part of the credit must be attributed to figures from the engineering community, such as Clough, Irons, Taig, etc., who were responsible for producing highly innovative numerical procedures that paved the way for the development of computer codes that could tackle problems of industrial magnitude with comparatively modest computational power.

The finite element method produces an approximation based on piecewise polynomial approximation on an underlying mesh. As with many classical numerical methods, the basic premise behind the finite element method is that improved accuracy is sought through reduction of the mesh-size h through mesh refinement (adaptive or otherwise) based on low order piecewise polynomial approximation.

In the mid-seventies, an alternative strategy was investigated by Szabó and his group in St. Louis, based on fixing the underlying mesh and seeking convergence by increasing the degree p of the polynomials used to construct the approximation. Numerical experiments on linear elasticity problems showed that this p-version approach was often superior to the h-version of the finite element method. In particular, for problems with smooth solutions, the rate of convergence appeared to be *exponential* while for problems with corner singularities, the rate of convergence appeared to be *double* that of the h-version on a sequence of quasi-uniform meshes. In the first theoretical analysis of the p-version, Babuška, Szabó and Katz (1981) proved these conclusions to be true. The analysis of the singular case exploits the fact that the solutions of linear elasticity problems have a specific structure that lends itself to approximation by polynomials, resulting in the doubling of the rate of convergence.

Encouraged by these results, investigations were made into how the advantages of h and p refinement might be combined to best effect in the so-called hp-version. The first theoretical analysis of the hp-version was given by Babuška and Dorr (1981). The exponential convergence of the hp-version for linear elliptic problems with piecewise analytic data was established in a series of landmark papers by Guo and Babuška. The key to the analysis again lies in the fact that the solutions of such problems have a specific structure that lends itself to efficient approximation by appropriately constructed hp-finite element spaces. In particular, this body of work included the introduction of so-called *countably normed spaces* and the demonstration that the solutions of linear elliptic problems with piecewise analytic data belong to such spaces. Approximation theoretic results and guidelines for the construction of the *hp* spaces were developed for approximation of these countably normed spaces culminating in the proof of the exponential convergence.

The success of *hp*-methods for the approximation of singularities that arise in elliptic problems has led to investigations, particularly by Schwab and coworkers, into what advantages the methodology might offer for the resolution of other local features such as boundary layers. Schwab and Suri (1996) analysed the performance of the h, p, and hp versions for a singularly perturbed two-point boundary value problem and showed the possibility of obtaining *robust exponential convergence*, where the constants and rate of convergence are independent of the perturbation parameter. This was later extended to include model problems in two and three dimensions by Schwab's group at Zürich, including convection dominated problems.

Another important area where the *hp*-methodology offers significant advantages includes numerical approximation of dimensionally reduced models for elastic plates, where the so-called *locking effects* can render low order approximations practically useless. Here, higher-order methods result in the alleviation of locking without the need to resort to numerical remedies such as reduced integration. These and other advantages of the *hp*-methodology have led to a number of commercial finite element codes such as MSC Nastran, proPHLEX, PolyFEM, Pro/MECHANICA, STRESSCHECK and STRIPE, now providing facilities for p and hp refinement.

The present book by Ch. Schwab is the first to discuss the theoretical aspects of *hp* finite element methods in depth. Chapter 1 deals with the variational formulation of boundary value problems and starts out innocently enough with a second order self-adjoint problem in one space dimension. Nevertheless, by the end of the 42-page chapter, the reader will have seen the generalised Lax–Milgram lemma and its application to a number of variational formulations of the bar problem, and met the trace spaces $H^{1/2}$ and $H^{1/2}_{00}$.

A brief introduction to the finite element method in Chapter 2, is followed by Chapter 3 (just under 100 pages) consisting of a detailed treatment of the h, p, and *hp*-version applied to problems in one space dimension. The coverage includes the approximation theory for singular and boundary layer solutions, mesh grading, and a posteriori error estimation. The chapter begins with an elementary discussion of the computation of element matrices, but one soon reaches a level found in a journal article. Indeed, the sections on boundary layers and convection diffusion problems are reproductions of the author's joint publications in these areas.

The reader would be well advised to thoroughly master the content of Chapter 3 before proceeding to the two dimensional case in Chapter 4. This is another substantial chapter of 67 pages that is chiefly concerned with establishing the exponential convergence of the *hp*-version for scalar elliptic problems, following in the footsteps of Babuška and Guo.

The application of *hp*-methods to elliptic systems forms the remainder of the book. Chapter 5 deals with mixed finite element methods for Stokes equations and has a full discussion of the standard background theory on the numerical treatment of saddle point problems and the Babuška–Brezzi condition. The chapter culminates with a proof of stability of various families of mixed p-finite elements. The final chapter deals with *hp*-methods for linear elasticity, with an emphasis on the modelling of thin domains and approximation of the plate and shell models that arise. An extended discussion of the locking phenomenon is given. The book concludes with two short appendixes on Sobolev spaces, Hilbert space interpolation, and orthogonal polynomials.

The book complements other texts in the area [1, 2] that are at a more elementary level and focus more on the practical implementation aspects. The manuscript is based on graduate lectures presented by the author to an audience of engineers and mathematicians at the ETH Zürich. In principle, the inclusion of background material means that the book should be accessible to a graduate student with quite a modest background in numerical analysis of elliptic partial differential equations. However, the demanding pace of the text would leave many UK graduate students in mathematics trailing in its wake. Exercises are included in the text, ranging from trivial computations to deeper applications of the theory.

My only real criticism of the book lies in the number of minor typographical errors and inconsistencies that should have easily been detected by a copy editor. At a cost of $85.00 for 374 pages, I would expect the publisher to produce a far more polished product. Nevertheless, this is a detailed and authoritative account of the theory of *hp*-version finite element methods at the end of the 1990s, and provides a much needed reference source for theoreticians in this area.

References

[1] B. Szabó and I. Babuška, *Finite Element Analysis*, John Wiley and Sons, Inc., 1991.
[2] G. Em. Karniadakis and S. Sherwin, *Spectral/hp Element Methods for CFD*, Oxford University Press, 1999.

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3[41A10, 42A10, 65M70, 65T10]—*Spectral methods in Matlab*, by Lloyd N. Trefethen, SIAM, Philadelphia, PA, 2000, xvi+165 pp., 23 1/2 cm, softcover, $36.00

This book is published within the series "Software, Environments, Tools"; in other words it is meant to be a "cookbook" for someone who is curious about learning spectral methods but does not want to go through a more comprehensive spectral method book or course, at least not at the beginning. It builds on the powerful Matlab platform and brings the essentials of spectral collacation methods with just forty short Matlab "M-files". These Matlab codes will also generate intriguing graphics to vividly illustrate the numerical results.

Spectral methods have been under rapid development in the last twenty-five years. There are many books written in this period, most notably the pioneering book by Gottlieb and Orszag in 1977 and the comprehensive book by Canuto, Hussaini, Quarteroni and Zang in 1988. The book under review is different from these comprehensive books. Although it does explain the essential background of spectral methods, in order to give the readers the basic ideas before letting them play with the Matlab codes, the emphasis here is clearly not on a comprehensive